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Stochastic Analysis of an Elastic 3D Half-Space Respond to Random Boundary Displacements: Exact Results and Karhunen-Loéve Expansion

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Abstract A stochastic response of an elastic 3D half-space to random displacement excitations on the boundary plane is studied. We derive exact results for the case of white noise excitations which are then used to give convolution representations for the case of general finite correlation length fluctuations of displacements prescribed on the boundary. Solutions to these elasticity problem are random fields which appear to be horizontally homogeneous but inhomogeneous in the vertical direction. This enables us to construct explicitly the Karhunen-Loève (K-L) series expansion by solving the eigen-value problem for the correlation operator. Simulation results are presented and compared with the exact representations derived for the displacement correlation tensor. This paper is a complete 3D generalization of the 2D case study we presented in Sabelfeld and Shalimova (J. Stat. Phys. 132(6):1071–1095, 2008).

Keywords White noise · Karhunen-Loeve expansion · Poisson integral formula · Boundary random excitations · 3D Lame equation

1 Introduction

Stochastic partial differential equations are known to be a very effective tool in modeling complicated phenomena in all fields of science and technology. Examples include wave propagation in random media [7], transport through highly heterogeneous porous media [1, 4, 6], randomly forced Navier-Stokes equation [3], etc. Many interesting examples can be found in material science [2, 13, 23], chemistry and biology [14, 24], as well as in cosmology (e.g., see [20]). Note that the input random fields can be considered both as

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a natural source of stochastic fluctuations, or as a model to describe extremely complicated irregularities and uncertainties (e.g., see [15, 22, 26]).

In electrical impedance tomography [9], an important problem is to evaluate a global response to random boundary excitations, and to estimate local fluctuations of the solution fields. Similar analysis is made in the inverse problems of elastography [12, 18], acoustic scattering from rough surfaces [28], and reaction-diffusion equations with white noise boundary perturbations [22]. An interesting application of the model we study in this paper is the analysis of the dislocations in crystals by X-ray diffraction [8]. The physical measurement method is based on the X-ray scattering from relaxed heteroepitaxial layers with the misfit dislocations randomly distributed at the interface between the layer and the substrate.

It should be noted that the cases where the fluctuations are governed by random coefficients of PDS, or their source terms, are widely used and intensively analyzed, while the random boundary conditions are not so well studied. The main reason is that in this case, we deal with *statistically inhomogeneous* random fields, hence the well known and commonly used spectral methods are here not applicable anymore. Another difficulty comes from the necessity to deal with boundary conditions and treat the relevant random boundary functions.

The main method for modeling inhomogeneous random fields is the Karhunen-Loève (K-L) expansion. Generally, it is computational demanding because it requires to solve numerically eigen-value problems of high dimension. However in some practically interesting cases models with analytically solvable eigen-value problem for the correlation operator can be obtained. This gives then a very efficient numerical method because as a rule, the K-L expansions are very fast convergent. We mention also the polynomial chaos expansion approach, a method in which it is attempted to reduce the original stochastic boundary value problem to a series of deterministic equations (e.g., see [25, 27]). This method however is applicable only if a small number of the series expansion is sufficient for a good approximation which is rather rare in practice.

In many interesting cases the solution of a PDE is a partially homogeneous random field generated by the homogeneous random excitations on the boundary. We analyzed the cases of Laplace, biharmonic, and Lamé equations in [17], the Stokes equation in [19], and the fractional Laplace equation in [20]. In [21] we have given exact representations for the correlation tensor, and the K-L expansions of the displacements in the case of the elastic half-plane. In this paper we extend these results to the half-space $D^+ = R^3_+$.

2 The System of Lamé Equations Governing an Elastic Half-Space

Let us consider the Dirichlet problem for the system of Lamé equations in the domain $D^+ \subset R^3$, the upper half-space with the boundary $\Gamma = \{(x, y, z) : z = 0\}$:

$$\Delta \mathbf{u}(\mathbf{x}) + \alpha \operatorname{grad} \operatorname{div} \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in D^+, \qquad \mathbf{u}(\mathbf{x}') = \mathbf{g}(\mathbf{x}') \quad \mathbf{x}' \in \Gamma = \partial D^+, \tag{1}$$

where $\mathbf{u}(\mathbf{x}) = (u_1(x, y, z), \dots, u_3(x, y, z))^T$ is a column vector of displacements, and $\mathbf{g}(\mathbf{x}') = (g_1(x', y'), \dots, g_3(x', y'))^T$ is the vector of displacements prescribed on the boundary. The elastic constant $\alpha = (\lambda + \mu)/\mu$ is expressed through the Lamé constants of elasticity λ and μ. We assume throughout the paper that these equations are properly written in a dimensionless form, so we deal with dimensionless displacements $\mathbf{u}$ as functions of dimensionless variable $\mathbf{x}$.

2.1 Poisson Formula for the Upper Half-Plane

The Poisson formula for the problem (1) has the form (see [11])

$$\mathbf{u}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x - x', y - y', z)\mathbf{g}(x', y')\, dx'\, dy', \tag{2}$$

where the matrix kernel K is given explicitly by

$$\begin{aligned} &K(x - x', y - y', z) \\ &= \frac{z}{2\pi r^3} \left\{ (1 - \beta)\mathrm{I} + \frac{3\beta}{r^2} \begin{pmatrix} (x - x')^2 & (x - x')(y - y') & (x - x')z \\ (x - x')(y - y') & (y - y')^2 & (y - y')z \\ (x - x')z & (y - y')z & z^2 \end{pmatrix} \right\}, \end{aligned} \tag{3}$$

where I is the identity matrix, $\beta = \frac{\lambda+\mu}{\lambda+3\mu}$, and

$$r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}.$$

3 Stochastic Boundary Value Problem

3.1 Correlation Tensor

Assume the boundary displacements g_i, $i = 1, 2, 3$ are homogeneous Gaussian random fields with zero mean, $\langle \mathbf{g} \rangle = 0$, then $\mathbf{u}(x, y, z)$ is also a Gaussian random field with $\langle \mathbf{u} \rangle = 0$. Hence this random field is uniquely defined by its correlation tensor. Note that there is no loss of generality since, generally, $\langle \mathbf{u} \rangle = \langle \mathbf{g} \rangle$ for homogeneous random field $\mathbf{g}$. This can be readily obtained by averaging of (2) and taking into account that the expectation of a homogeneous random field is a constant.

By the formula (2) for $\mathbf{u}$, the correlation tensor $B_u(\mathbf{x}_1; \mathbf{x}_2) = B_u(x_1, y_1, z_1; x_2, y_2, z_2)$ for the displacements can be written as follows

$$\begin{aligned} B_u(\mathbf{x}_1; \mathbf{x}_2) &= \langle \mathbf{u}(x_1, y_1, z_1) \otimes \mathbf{u}(x_2, y_2, z_2) \rangle \\ &= \int_{R^4} K(x_1 - x_1', y_1 - y_1', z_1) B_g(\mathbf{x}_1'; \mathbf{x}_2') K^T(x_2 - x_2', y_2 - y_2', z_2)\, d\mathbf{x}_1'\, d\mathbf{x}_2'. \end{aligned} \tag{4}$$

We use here the notation $\otimes$ for the direct product of vectors $\mathbf{u}(x_1, y_1, z_1)$ and $\mathbf{u}(x_2, y_2, z_2)$, and $B_g(\mathbf{x}_1'; \mathbf{x}_2')$ for the correlation tensor of the random boundary vector field $\mathbf{g}$

$$B_g(\mathbf{x}_1'; \mathbf{x}_2') = B_g(x_1', y_1'; x_2', y_2') = \langle \mathbf{g}(x_1', y_1') \otimes \mathbf{g}(x_2', y_2') \rangle.$$

Let us consider the case when $\mathbf{g}$ is a $2D$ white noise defined on the plane $z = 0$. This implies that

$$\{B_g(\mathbf{x}_1'; \mathbf{x}_2')\}_{ij} = \delta_{ij}\delta(x_1' - x_2')\delta(y_1' - y_2'), \quad i, j = 1, 2, 3.$$

Here we use standard notations, δ_{ij} for the Kronecker symbol, and $\delta(\cdot)$ for the Dirac δ-function.

Theorem 1 *The solution of the boundary value problem* (1) *with the prescribed Gaussian white noise displacements on the boundary is a Gaussian random field, horizontally homogeneous, and hence is uniquely defined by its correlation tensor which depends on the difference of the horizontal coordinates* $x_1 - x_2$ *and* $y_1 - y_2$, *while in the vertical direction, it depends on the product* $z_1 z_2$, *and* $z_1 \pm z_2$, *and has the following explicit form*:

$$B_u = \frac{z_1+z_2}{2\pi \hat{r}^3}\left\{(1-\beta)\mathrm{I} + \frac{3\beta}{\hat{r}^2}\begin{pmatrix} \tau_x^2 & \tau_x\tau_y & \tau_x(z_1-z_2) \\ \tau_x\tau_y & \tau_y^2 & \tau_y(z_1-z_2) \\ \tau_x(z_1-z_2) & \tau_y(z_1-z_2) & (z_1+z_2)^2 \end{pmatrix} + \frac{6\beta^2 z_1 z_2}{\hat{r}^4} \right.$$
$$\left. \times \begin{pmatrix} \hat{r}^2 - 5\tau_x^2 & -5\tau_x\tau_y & \tau_x(5(z_1+z_2) - \frac{\hat{r}^2}{z_1+z_2}) \\ -5\tau_x\tau_y & \hat{r}^2 - 5\tau_y^2 & \tau_y(5(z_1+z_2) - \frac{\hat{r}^2}{z_1+z_2}) \\ \tau_x(\frac{\hat{r}^2}{z_1+z_2} - 5(z_1+z_2)) & \tau_y(\frac{\hat{r}^2}{z_1+z_2} - 5(z_1+z_2)) & 2\hat{r}^2 - 5(\tau_x^2+\tau_y^2) \end{pmatrix}\right\}, \tag{5}$$

where

$$\hat{r} = \sqrt{\tau_x^2 + \tau_y^2 + (z_1+z_2)^2}, \quad \tau_x = x_1 - x_2, \tau_y = y_1 - y_2.$$

Proof For the white noise the formula (4) takes the form

$$B_u(\mathbf{x}_1;\mathbf{x}_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x_1 - x_1', y_1 - y_1', z_1) K^T(x_2 - x_1', y_2 - y_1', z_2)\, dx_1'\, dy_1'.$$

To integrate the right-hand side we use the Fourier transformation. Let us take a change of variables, $w_x = x_1' - x_2$ and $w_y = y_1' - y_2$, and then use new variables $\tau_x = x_1 - x_2$, $\tau_y = y_1 - y_2$. This yields

$$B_u(\tau_x, \tau_y; z_1, z_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\tau_x - w_x, \tau_y - w_y, z_1) K^T(-w_x, -w_y, z_2)\, dw_x\, dw_y.$$

The last formula has a convolution form and can be written shortly as

$$B_u(\tau_x, \tau_y; z_1, z_2) = K(\tau_x, \tau_y, z_1) * K(-w_x, -w_y, z_2).$$

The Fourier transform property for convolutions yields

$$F^{-1}[B_u] = F^{-1}[K(\tau_x, \tau_y, z_1)]F^{-1}[K(-w_x, -w_y, z_2)].$$

So we have to find the inverse transform $F^{-1}[K](\tau_x, \tau_y, z)$. Using the Fourier transform formulae (31)–(35) presented in Appendix A, we find that $F^{-1}[K](\tau_x, \tau_y, z) = e^{-z\sqrt{\xi_x^2+\xi_y^2}} G(\xi_x, \xi_y, z)$, and

$$F^{-1}[B_u] = e^{-(z_1+z_2)\sqrt{\xi_x^2+\xi_y^2}} G(\xi_x, \xi_y, z_1) G^*(\xi_x, \xi_y, z_2), \tag{6}$$

where the star sign stands for the complex conjugate transpose, and

$$G(\xi_x,\xi_y,z)=\mathrm{I}-\beta z\begin{pmatrix}\frac{\xi_x^2}{\sqrt{\xi_x^2+\xi_y^2}} & \frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}} & \imath\xi_x\\ \frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}} & \frac{\xi_y^2}{\sqrt{\xi_x^2+\xi_y^2}} & \imath\xi_y\\ \imath\xi_x & \imath\xi_y & -\sqrt{\xi_x^2+\xi_y^2}\end{pmatrix}. \tag{7}$$

Note that we have taken the inverse Fourier transform of the correlation tensor with respect to the variables x, y. It means that we get a partial spectral tensor S_u. Indeed, by definition

$$\begin{aligned}S_u(\xi_x,\xi_y,z_1,z_2) &= F^{-1}[B_u(\tau_x,\tau_y,z_1,z_2)]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathrm{i}(\xi_x\tau_x+\xi_y\tau_y)}B_u(\tau_x,\tau_y,z_1,z_2)\,d\tau_x\,d\tau_y,\end{aligned}$$

hence

$$S_u(\xi_x,\xi_y,z_1,z_2)=e^{-\sqrt{\xi_x^2+\xi_y^2}(z_1+z_2)}G(\xi_x,\xi_y,z_1)G^*(\xi_x,\xi_y,z_2). \tag{8}$$

We will find now the correlation tensor B_u by using the relevant Fourier transform properties. To this end we rewrite (6) as follows:

$$\begin{aligned}F^{-1}[B_u] &= e^{-(z_1+z_2)\sqrt{\xi_x^2+\xi_y^2}}G(\xi_x,\xi_y,z_1)G^*(\xi_x,\xi_y,z_2)\\ &= e^{-(z_1+z_2)\sqrt{\xi_x^2+\xi_y^2}}\left\{\mathrm{I}-\beta(z_1+z_2)\begin{pmatrix}\frac{\xi_x^2}{\sqrt{\xi_x^2+\xi_y^2}} & \frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}} & \imath\frac{\xi_x(z_1-z_2)}{z_1+z_2}\\ \frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}} & \frac{\xi_y^2}{\sqrt{\xi_x^2+\xi_y^2}} & \imath\frac{\xi_y(z_1-z_2)}{z_1+z_2}\\ \imath\frac{\xi_x(z_1-z_2)}{z_1+z_2} & \imath\frac{\xi_y(z_1-z_2)}{z_1+z_2} & -\sqrt{\xi_x^2+\xi_y^2}\end{pmatrix}\right.\\ &\quad\left.+2\beta^2 z_1z_2\begin{pmatrix}\xi_x^2 & \xi_x\xi_y & -\imath\xi_x\sqrt{\xi_x^2+\xi_y^2}\\ \xi_x\xi_y & \xi_y^2 & -\imath\xi_y\sqrt{\xi_x^2+\xi_y^2}\\ \imath\xi_x\sqrt{\xi_x^2+\xi_y^2} & \imath\xi_y\sqrt{\xi_x^2+\xi_y^2} & \xi_x^2+\xi_y^2\end{pmatrix}\right\}.\end{aligned} \tag{9}$$

To obtain the desired representation for the tensor B_u, we convert (9) by using the formulae (36)–(39) derived in Appendix A. □

3.2 Spectral Representations for Partially Homogeneous Random Fields

So as follows from Theorem 1, the solution random field $\mathbf{u}(x,y,z)$ is homogeneous with respect to the variables x, y, it means that

$$B_u=\langle\mathbf{u}(x_1,y_1,z_1)\otimes\mathbf{u}(x_2,y_2,z_1)\rangle=B_u(x_1-x_2,y_1-y_2,z_1,z_2).$$

As mentioned above, the random fields with this property are called partially homogeneous, with the partial spectral tensor $S_u(\xi_x,\xi_y,z_1,z_2)$ given by (8).

To simulate partially homogeneous random fields $\mathbf{u}(\mathbf{x})$, the randomization spectral model described in [21] can be used. This model first presented in [16] has the form

$$\hat{\mathbf{u}}(x,y,z) = \frac{1}{[p(\xi_x,\xi_y)]^{1/2}}\big[\boldsymbol{\zeta}_\xi(z)\cos(\xi_x x+\xi_y y)+\boldsymbol{\eta}_\xi(z)\sin(\xi_x x+\xi_y y)\big], \tag{10}$$

where the random variables ξ_x, ξ_y have a distribution density $p(\xi_x,\xi_y)$ in the wave space (which can be chosen quite arbitrarily), and the real-valued 6-dimensional field $(\boldsymbol{\zeta}_\xi(z), \boldsymbol{\eta}_\xi(z))^T$ for fixed ξ_x, ξ_y has the correlation tensor

$$\begin{aligned} B_{(\zeta,\eta)}(z_1,z_2) &= \begin{pmatrix} \langle\boldsymbol{\zeta}_\xi(z_1)\otimes\boldsymbol{\zeta}_\xi(z_2)\rangle & \langle\boldsymbol{\zeta}_\xi(z_1)\otimes\boldsymbol{\eta}_\xi(z_2)\rangle \\ \langle\boldsymbol{\eta}_\xi(z_1)\otimes\boldsymbol{\zeta}_\xi(z_2)\rangle & \langle\boldsymbol{\eta}_\xi(z_1)\otimes\boldsymbol{\eta}_\xi(z_2)\rangle \end{pmatrix} \\ &= \begin{pmatrix} \Re S_u(\cdot,z_1,z_2) & \Im S_u(\cdot,z_1,z_2) \\ -\Im S_u(\cdot,z_1,z_2) & \Re S_u(\cdot,z_1,z_2) \end{pmatrix}. \end{aligned} \tag{11}$$

Here we use the notation $\Re S_u$ and $\Im S_u$ for the real and imaginary part of S_u, respectively. Using the decomposition of S_u in the product (8) it is easy to verify that

$$\begin{pmatrix} \Re S_u & \Im S_u \\ -\Im S_u & \Re S_u \end{pmatrix} = e^{-\sqrt{\xi_x^2+\xi_y^2}(z_1+z_2)} \begin{pmatrix} \Re G & \Im G \\ -\Im G & \Re G \end{pmatrix}_{(\cdot,z_1)} \begin{pmatrix} \Re G & \Im G \\ -\Im G & \Re G \end{pmatrix}^T_{(\cdot,z_2)}.$$

Then the 6-dimensional vector field $(\boldsymbol{\zeta}_\xi(z), \boldsymbol{\eta}_\xi(z))^T$ defined by

$$\begin{pmatrix} \boldsymbol{\zeta}_\xi \\ \boldsymbol{\eta}_\xi \end{pmatrix} = e^{-z\sqrt{\xi_x^2+\xi_y^2}} \begin{pmatrix} \Re G & \Im G \\ -\Im G & \Re G \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\eta} \end{pmatrix}, \tag{12}$$

has the desired correlation tensor (11). Here $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ are independent 3-dimensional Gaussian random vectors with zero mean and unit covariance matrix.

Thus we have a Randomization spectral model of type (10) where the random vectors $\boldsymbol{\zeta}_\xi$ and $\boldsymbol{\eta}_\xi$ are constructed by (12), and ξ_x, ξ_y are sampled according to an arbitrary density p in the wave space.

This model has the desired correlation tensor, i.e., $B_{\hat{u}} = B_u$,

$$\begin{aligned} B_u(\tau_x,\tau_y,z_1,z_2) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}(\xi_x\tau_x+\xi_y\tau_y)} S_u(\xi_x,\xi_y,z_1,z_2)\,d\xi_x\,d\xi_y \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[\Re S_u\cos(\xi_x\tau_x+\xi_y\tau_y)-\Im S_u\sin(\xi_x\tau_x+\xi_y\tau_y)\big]\,d\xi_x\,d\xi_y, \end{aligned} \tag{13}$$

here $\tau_x = x_1 - x_2$, $\tau_y = y_1 - y_2$ (see [21]).

Concerning the sampling of the wave vectors, one of the simplest choice is a uniform distribution. Then however we have to cut-off the range where the wave numbers ξ_x and ξ_y are defined, say from $-R_1$ to R_1 and $-R_2$ to R_2, R_i being large enough. In addition, to ensure that all the high-dimensional distributions of the model are close to Gaussian, one usually takes a sum of independent realizations of models (10). In another version, one makes a partition of the wave number space into bins, and takes a sum of samples with wave number modes sampled independently within each bin (e.g., see [10, 16]).

This is generally different from a standard cubature-approximation of the stochastic integral representation of the random field with the correlation tensor (13) where the integration is taken from $-R_1$ to R_1 and $-R_2$ to R_2. This leads to an approximation in the form

$$\mathbf{u}(x,y,z) \approx \hat{\mathbf{u}}(x,y,z) = \frac{1}{2\sqrt{R_1 R_2}} \sum_{\substack{k,m=-\infty \\ (k,m)\neq(0,0)}}^{\infty} e^{-\pi z\sqrt{(k/R_1)^2+(m/R_2)^2}}$$
$$\times \Big[\big(\Re G(k,m,z)\boldsymbol{\zeta}_{k,m} + \Im G(k,m,z)\boldsymbol{\eta}_{k,m}\big)\cos\pi\left(\frac{kx}{R_1}+\frac{my}{R_2}\right)$$
$$+ \big(\Re G(k,m,z)\boldsymbol{\eta}_{k,m} - \Im G(k,m,z)\boldsymbol{\zeta}_{k,m}\big)\sin\pi\left(\frac{kx}{R_1}+\frac{my}{R_2}\right)\Big]$$

where $\boldsymbol{\eta}_{k,m}$, $\boldsymbol{\zeta}_{k,m}$ are families of independent standard Gaussian vectors, and $G(k,m,z)$ is the matrix G defined in (7) with the values $\xi_k = \pi k/R_1$, $\xi_m = \pi m/R_2$:

$$G(k,m,z) = \mathrm{I} - \beta z\pi \begin{pmatrix} \frac{k^2}{R_1^2\sqrt{(k/R_1)^2+(m/R_2)^2}} & \frac{km}{R_1R_2\sqrt{(k/R_1)^2+(m/R_2)^2}} & \imath\frac{k}{R_1} \\ \frac{km}{R_1R_2\sqrt{(k/R_1)^2+(m/R_2)^2}} & \frac{m^2}{R_2^2\sqrt{(k/R_1)^2+(m/R_2)^2}} & \imath\frac{m}{R_2} \\ \imath\frac{k}{R_1} & \imath\frac{m}{R_2} & -\sqrt{(\frac{k}{R_1})^2+(\frac{m}{R_2})^2} \end{pmatrix}. \tag{14}$$

This model has a correlation tensor which is an approximation to the original correlation tensor B_u:

$$B_u(\tau_x,\tau_y,z_1,z_2) \approx \frac{1}{4R_1R_2} \sum_{\substack{k,m=-\infty \\ (k,m)\neq(0,0)}}^{\infty} e^{-\pi(z_1+z_2)\sqrt{(k/R_1)^2+(m/R_2)^2}}$$
$$\times \Big[\Re\{G(k,m,z)G^*(k,m,z)\}\cos\pi\left(\frac{k\tau_x}{R_1}+\frac{m\tau_y}{R_2}\right)$$
$$- \Im\{G(k,m,z)G^*(k,m,z)\}\sin\pi\left(\frac{k\tau_x}{R_1}+\frac{m\tau_y}{R_2}\right)\Big].$$

All these arguments are basically rigorous and use essentially the important properties that (1) the solution random field is partially homogeneous, and (2) the partial spectral tensor $S_u(\cdot,z_1,z_2)$ can be represented as a product of two matrices, $G(\cdot,z_1)$ and $G^*(\cdot,z_2)$.

In the next section we treat the solution as a general inhomogeneous random field, and rigorously derive the Karhunen-Loève expansion for the random field itself, and for its correlation tensor.

3.3 The Karhunen-Loève Expansion

The Karhunen-Loève expansion has the form (e.g., see [17, 21, 29]):

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\eta_k \mathbf{h}_k(\mathbf{x}),$$

where η_k is a family of independent random variables, λ_k and $\mathbf{h}_k(\mathbf{x})$ are the eigen-values and eigen-functions of the covariance operator B_u, i.e.,

$$\int B_u(\mathbf{x}_1, \mathbf{x}_2)\mathbf{h}_k(\mathbf{x}_2)\, d\mathbf{x}_2 = \lambda_k \mathbf{h}_k(\mathbf{x}_1).$$

In our case $\mathbf{u}$ is partially homogeneous, that means, it is homogeneous with respect to the variables x, y, and is inhomogeneous with respect to z. It implies, that the eigen-value problem reads

$$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty B_u(x_1 - x_2, y_1 - y_2, z_1, z_2)\mathbf{h}_k(x_2, y_2, z_2)\, dx_2\, dy_2\, dz_2 = \lambda_k \mathbf{h}_k(x_1, y_1, z_1). \tag{15}$$

For the correlation tensor the Karhunen-Loève expansion looks like

$$B_u(x_1 - x_2, y_1 - y_2, z_1, z_2) = \sum_{k=1}^{\infty} \lambda_k (\mathbf{h}_k(x_1, y_1, z_1) \otimes \mathbf{h}_k(x_2, y_2, z_2)).$$

For our domain D^+ we apply a cut-off integration for (15) from $-R_1$ to R_1 over the variable x and from $-R_2$ to R_2 over y, i.e., we solve the eigen-value problem

$$\int_0^\infty \int_{-R_1}^{R_2} \int_{-R_2}^{R_2} B_u(x_2 - x_1, y_1 - y_2, z_1, z_2)\mathbf{h}_k(x_2, y_2, z_2)\, dx_2\, dy_2\, dz_2 = \lambda_k \mathbf{h}_k(x_1, y_1, z_1) \tag{16}$$

where R_i are sufficiently large. In what follows and throughout the paper we preserve for simplicity the notation $\mathbf{u} = (u_1, u_2, u_3)^T$ and B_u for the problem with the introduced cut-off, that means the problem (1) is considered in the region $\{(x, y, z) : -R_1 \le x \le R_1, -R_2 \le y \le R_2, z > 0\}$.

Theorem 2 *The solution random field* $\mathbf{u}(x, y, z)$ *has the following Karhunen-Loève expansion*

$$\begin{pmatrix} u_1(x, y, z) \\ u_2(x, y, z) \\ u_3(x, y, z) \end{pmatrix} = \sum_{\substack{k,m=-\infty \\ (k,m)\neq(0,0)}}^{\infty} \sum_{i=1}^{3} \frac{e^{-\pi z\sqrt{(k/R_1)^2+(m/R_2)^2}}}{2\sqrt{R_1 R_2}} \begin{pmatrix} a_i(\zeta^i_{k,m}\cos\gamma_{km} + \tilde{\zeta}^i_{k,m}\sin\gamma_{km}) \\ b_i(\zeta^i_{k,m}\cos\gamma_{km} + \tilde{\zeta}^i_{k,m}\sin\gamma_{km}) \\ c_i(\zeta^i_{k,m}\sin\gamma_{km} - \tilde{\zeta}^i_{k,m}\cos\gamma_{km}) \end{pmatrix},$$

where $\zeta^i_{k,m}$, $\tilde{\zeta}^i_{k,m}$ *are independent standard Gaussian random variables,* $\gamma_{km} = \pi(\frac{kx}{R_1} + \frac{my}{R_2})$, *and the coefficients* a_i, b_i, c_i *are given explicitly by*

$$a_1 = 1 - \frac{\beta z\pi k^2}{R_1^2\sqrt{(k/R_1)^2 + (m/R_2)^2}}, \qquad b_1 = -\frac{\beta z\pi km}{R_1 R_2\sqrt{(k/R_1)^2 + (m/R_2)^2}},$$

$$c_1 = \beta z\pi \frac{k}{R_1},$$

$$a_2 = -\frac{\beta z\pi km}{R_1 R_2\sqrt{(k/R_1)^2 + (m/R_2)^2}}, \qquad b_2 = 1 - \frac{\beta z\pi m^2}{R_2^2\sqrt{(k/R_1)^2 + (m/R_2)^2}},$$

$$c_2 = \beta z\pi \frac{m}{R_2},$$

$$a_3 = \beta z\pi k/R_1, \qquad b_3 = \beta z\pi m/R_2, \qquad c_3 = -1 - \beta z\pi\sqrt{(k/R_1)^2 + (m/R_2)^2}.$$

The correlation tensor is represented by the series

$$B_u = \sum_{\substack{k,m=-\infty \\ (k,m)\neq(0,0)}}^{\infty} \frac{e^{-\pi(z_1+z_2)\sqrt{(k/R_1)^2+(m/R_2)^2}}}{4R_1R_2} \Bigg\{ \Bigg(\mathrm{I} + 2\pi^2\beta^2 z_1 z_2 \begin{pmatrix} \frac{k^2}{R_1^2} & \frac{km}{R_1R_2} & 0 \\ \frac{km}{R_1R_2} & \frac{m^2}{R_2^2} & 0 \\ 0 & 0 & \frac{k^2}{R_1^2}+\frac{m^2}{R_2^2} \end{pmatrix}$$
$$- \frac{\pi\beta(z_1+z_2)}{\sqrt{(k/R_1)^2+(m/R_2)^2}} \begin{pmatrix} \frac{k^2}{R_1^2} & \frac{km}{R_1R_2} & 0 \\ \frac{km}{R_1R_2} & \frac{m^2}{R_2^2} & 0 \\ 0 & 0 & -\frac{k^2}{R_1^2}+\frac{m^2}{R_2^2} \end{pmatrix} \Bigg) \cos\hat{\gamma}_{km}$$
$$+ \Bigg(\beta\pi(z_1-z_2) \begin{pmatrix} 0 & 0 & \frac{k}{R_1} \\ 0 & 0 & \frac{m}{R_2} \\ \frac{k}{R_1} & \frac{m}{R_2} & 0 \end{pmatrix}$$
$$+ 2\beta^2 z_1 z_2 \pi^2 \sqrt{\frac{k^2}{R_1^2}+\frac{m^2}{R_2^2}} \begin{pmatrix} 0 & 0 & \frac{k}{R_1} \\ 0 & 0 & \frac{m}{R_2} \\ \frac{-k}{R_1} & \frac{-m}{R_2} & 0 \end{pmatrix} \Bigg) \sin\hat{\gamma}_{km} \Bigg\} \tag{17}$$

where $\hat{\gamma}_{km} = \pi(\frac{k\tau_x}{R_1} + \frac{m\tau_y}{R_2})$.

Proof The proof and the relevant series expansions will immediately follow from the solution of the eigen-value problem for the correlation tensor (16). To get the Karhunen-Loève expansions for $\mathbf{u}$ we split it into three independent random fields:

$$\mathbf{u}(x,y,z) = \mathbf{V}_1(x,y,z) + \mathbf{V}_2(x,y,z) + \mathbf{V}_3(x,y,z). \tag{18}$$

Since $\mathbf{V}_i$ and $\mathbf{V}_j$ are independent ($i \neq j$), the correlation tensor can be represented in the form

$$B_u = \langle \mathbf{u}(\mathbf{x}_1)\otimes\mathbf{u}(\mathbf{x}_2)\rangle = \langle \mathbf{V}_1(\mathbf{x}_1)\otimes\mathbf{V}_1(\mathbf{x}_2)\rangle + \langle \mathbf{V}_2(\mathbf{x}_1)\otimes\mathbf{V}_2(\mathbf{x}_2)\rangle + \langle \mathbf{V}_3(\mathbf{x}_1)\otimes\mathbf{V}_3(\mathbf{x}_2)\rangle, \tag{19}$$

or, shortly, $B_u = B_{V_1} + B_{V_2} + B_{V_3}$, where $(\mathbf{x}_j) = (x_j, y_j, z_j)$ for $j = 1, 2, 3$. So we have to solve the eigen-value problems for the correlation tensors B_{V_i}

$$\int_0^\infty \int_{-R_1}^{R_1} \int_{-R_2}^{R_2} B_{V_i}(x_2-x_1, y_1-y_2, z_1, z_2) h^i_{k,m}(x_2, y_2, z_2)\, dx_2\, dy_2\, dz_2 = \lambda^i_{k,m} h^i_{k,m}(x_1, y_1, z_1), \tag{20}$$

for $i = 1, 2, 3$ and $k, m = 0, \pm1, \pm2, \ldots, (k, m) \neq (0, 0)$.

In the following statement we solve these three eigen-value problems.

Lemma 1 *The eigen-value problems* (20) *have the following systems of eigen-values* $\lambda^i_{k,m}$, $\widetilde{\lambda}^i_{k,m}$ *and the corresponding eigen-functions,*

$$h^i_{k,m}(\mathbf{x}) = \frac{e^{-z\sqrt{\xi_k^2+\xi_m^2}}}{\Delta_i}\begin{pmatrix} a_i\cos(x\xi_k + y\xi_m)\\ b_i\cos(x\xi_k + y\xi_m)\\ c_i\sin(x\xi_k + y\xi_m)\end{pmatrix},$$

$$\widetilde{h}^i_{k,m}(\mathbf{x}) = \frac{e^{-z\sqrt{\xi_k^2+\xi_m^2}}}{\Delta_i}\begin{pmatrix} a_i\sin(x\xi_k + y\xi_m)\\ b_i\sin(x\xi_k + y\xi_m)\\ -c_i\cos(x\xi_k + y\xi_m)\end{pmatrix}$$

with

$$a_1 = 1 - \frac{z\beta\xi_k^2}{\sqrt{\xi_k^2+\xi_m^2}}, \qquad b_1 = -\frac{z\beta\xi_k\xi_m}{\sqrt{\xi_k^2+\xi_m^2}}, \qquad c_1 = z\beta\xi_k,$$

$$a_2 = -\frac{z\beta\xi_k\xi_m}{\sqrt{\xi_k^2+\xi_m^2}}, \qquad b_2 = 1 - \frac{z\beta\xi_m^2}{\sqrt{\xi_k^2+\xi_m^2}}, \qquad c_2 = z\beta\xi_m,$$

$$a_3 = z\beta\xi_k, \qquad b_3 = z\beta\xi_m, \qquad c_3 = -1 - z\beta\sqrt{\xi_k^2+\xi_m^2},$$

and

$$\Delta_1^2 = \frac{R_1R_2}{\sqrt{\xi_k^2+\xi_m^2}}\left[1 - \frac{(1-\beta)\beta\xi_k^2}{(\xi_k^2+\xi_m^2)}\right],$$

$$\Delta_2^2 = \frac{R_1R_2}{\sqrt{\xi_k^2+\xi_m^2}}\left[1 - \frac{(1-\beta)\beta\xi_m^2}{(\xi_k^2+\xi_m^2)}\right], \qquad \Delta_3^2 = \frac{R_1R_2(1+\beta+\beta^2)}{\sqrt{\xi_k^2+\xi_m^2}}.$$

Here the subindexes $_i$ *stand for the* i*-th series of eigen-functions.*

Proof The vectors $h^i_{k,m}$, $h^i_{j,l}$ are pairwise orthogonal, i.e.,

$$\int_{-R_1}^{R_1}\int_{-R_2}^{R_2}(h^i_{k,m}(\mathbf{x}), h^i_{j,l}(\mathbf{x}))\,dy\,dx = \delta_{kj}\delta_{ml}$$

for all $k, m, j, l = 0, \pm1, \pm2, \ldots$, but $(k,m) \neq (0,0)$, $(j,l) \neq (0,0)$ and $i = 1,2,3$, as well as $\widetilde{h}^i_{k,m}$, $\widetilde{h}^i_{j,l}$. Here δ_{kl} is the Kronecker symbol. The vectors $h^i_{k,m}$ and $\widetilde{h}^i_{k,m}$ are orthogonal, too. The normalization follows from

$$\begin{aligned}\|h^1_{k,m}\|^2 &= \frac{1}{\Delta_1^2}\int_0^\infty\int_{-R_1}^{R_1}\int_{-R_2}^{R_2}(h^1_{k,m}(\mathbf{x}), h^1_{k,m}(\mathbf{x}))\,d\mathbf{x}\\ &= \frac{1}{\Delta_1^2}\int_0^\infty\int_{-R_1}^{R_1}\int_{-R_2}^{R_2} e^{-2z\sqrt{\xi_k^2+\xi_m^2}}\\ &\quad\times\left[(a_1^2+b_1^2)\cos^2(x\xi_k+y\xi_m) + c_1^2\sin^2(x\xi_k+y\xi_m)\right]d\mathbf{x}\end{aligned}$$

$$= \frac{2R_1R_2}{\Delta_1^2}\int_0^\infty \left(1+2\beta^2\xi_k^2z^2 - \frac{2\beta\xi_k^2 z}{\sqrt{\xi_k^2+\xi_m^2}}\right)e^{-2z\sqrt{\xi_k^2+\xi_m^2}}\,dz$$

$$= \frac{2R_1R_2}{2\sqrt{\xi_k^2+\xi_m^2}\Delta_1^2}\left(1-\frac{(1-\beta)\beta\xi_k^2}{\xi_k^2+\xi_m^2}\right) = 1,$$

since

$$\int_{-R_1}^{R_1}\int_{-R_2}^{R_2}\cos^2(x\xi_k+y\xi_m)\,dx\,dy = \int_{-R_1}^{R_1}\int_{-R_2}^{R_2}\sin^2(x\xi_k+y\xi_m)\,dx\,dy = 2R_1R_2.$$

Note that $\|h_{k,m}^1\|^2 = \|\widetilde{h}_{k,m}^1\|^2$. Similar evaluations yield $\|h_{k,m}^i\|^2 = \|\widetilde{h}_{k,m}^i\|^2 = 1$ for $i = 2, 3$.

Now let us consider the eigen-value problem (20) for the tensors B_{V_i}. Let us introduce complex-valued vectors $H_{k,m}^i$ by $H_{k,m}^i = h_{k,m}^i + \imath\widetilde{h}_{k,m}^i$, so that

$$H_{k,m}^i(x,y,z) = e^{-z\sqrt{\xi_k^2+\xi_m^2}}e^{\imath(\xi_k x+\xi_m y)}\begin{pmatrix} a_i(z,k,m)\\ b_i(z,k,m)\\ -\imath c_i(z,k,m)\end{pmatrix}.$$

Since $\lambda_{k,m}^i = \widetilde{\lambda}_{k,m}^i$, we can rewrite (20) in the form

$$\int_0^\infty\int_{-R_1}^{R_1}\int_{-R_2}^{R_2} B_{V_i}(x_1-x_2, y_1-y_2, z_1, z_2)H_{k,m}^i(x_2,y_2,z_2)\,dx_2\,dy_2\,dz_2$$
$$= \lambda_{k,m}^i H_{k,m}^i(x_1,y_1,z_1).$$

Let us first consider this eigen-value problem for B_{V_1}. Substituting $H_{k,m}^1$ we find that

$$\int_0^\infty\int_{-R_1}^{R_1}\int_{-R_2}^{R_2} e^{-\imath(\xi_k\tau_x+\xi_m\tau_y)}B_{V_1}(\tau_x,\tau_y,z_1,z_2)\,dx_2\,dy_2$$
$$\times\begin{pmatrix} a_1(z_2,k,m)\\ b_1(z_2,k,m)\\ -\imath c_1(z_2,k,m)\end{pmatrix}e^{-(z_2-z_1)\sqrt{\xi_k^2+\xi_m^2}}\,dz_2$$
$$= \lambda_{k,m}^1\begin{pmatrix} a_1(z_1,k,m)\\ b_1(z_1,k,m)\\ -\imath c_1(z_1,k,m)\end{pmatrix}. \tag{21}$$

We notice that the inner integral is an approximation to the relevant spectral tensor

$$S_{V_1}(\xi_x,\xi_y,z_1,z_2) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\imath(\xi_x\tau_x+\xi_y\tau_y)}B_{V_1}(\tau_x,\tau_y,z_1,z_2)\,d\tau_x\,d\tau_y.$$

We use the approximation

$$S_{V_1}(\xi_x,\xi_y,z_1,z_2) \approx \widehat{S}_{V_1}(\xi_x,\xi_y,z_1,z_2),$$

where

$$\widehat{S}_{V_1}(\xi_x,\xi_y,z_1,z_2) = \frac{1}{2\pi}\int_{-R_1}^{R_1}\int_{-R_2}^{R_2} e^{-\imath(\xi_x,\tau_x+\xi_y\tau_y)}B_{V_1}(\tau_x,\tau_y,z_1,z_2)\,d\tau_x\,d\tau_y.$$

At the points $\xi_x = \xi_k$, $\xi_y = \xi_m$,

$$\widehat{S}_{V_1}(\xi_k, \xi_m, z_1, z_2) = \frac{1}{2\pi} \int_{-R_1}^{R_1} \int_{-R_2}^{R_2} e^{-\imath(\xi_k \tau_x + \xi_m \tau_y)} B_{V_1}(\tau_x, \tau_y, z_1, z_2)\, d\tau_x\, d\tau_y.$$

In what follow we will write for brevity S_{V_1} instead of $\widehat{S}_{V_1}$. The spectral tensor S_{V_1} has the form

$$S_{V_1}(\xi_k, \xi_m, z_1, z_2)$$
$$= e^{-(z_1+z_2)\rho} \begin{pmatrix} 1 + \beta^2 z_1 z_2 \frac{\xi_k^4}{\rho^2} - \beta(z_1+z_2)\frac{\xi_k^2}{\rho} & (\beta z_1 \frac{\xi_k^2}{\rho} - 1)\beta z_2 \frac{\xi_k \xi_m}{\rho} & \imath(1 - \beta z_1 \frac{\xi_k^2}{\rho})\beta z_2 \xi_k \\ \beta z_1 \frac{\xi_k \xi_m}{\rho}(\beta z_2 \frac{\xi_k^2}{\rho} - 1) & \beta^2 z_1 z_2 \frac{\xi_k^4}{\rho^2} & -\imath \beta^2 z_1 z_2 \frac{\xi_k^2 \xi_m}{\rho} \\ -\imath(1 - \beta z_2 \frac{\xi_k^2}{\rho})\beta z_1 \xi_k & \imath \beta^2 z_1 z_2 \frac{\xi_k \xi_m^2}{\rho} & -\beta^2 z_1 z_2 \xi_k^2 \end{pmatrix},$$

where we use the notation $\sqrt{\xi_k^2 + \xi_m^2} = \rho$.

We decompose S_{V_1} as $S_{V_1} = e^{-(z_1+z_2)\sqrt{\xi_k^2+\xi_m^2}} G(\xi_k, \xi_m, z_1) G_1(\xi_k, \xi_m, z_2)$ where $G(\xi_k, \xi_m, z_1)$ is defined by (7), and

$$G_1(\xi_k, \xi_m, z_2) = \begin{pmatrix} 1 - \beta z_2 \frac{\xi_k^2}{\sqrt{\xi_k^2+\xi_m^2}} & -\beta z_2 \frac{\xi_k \xi_m}{\sqrt{\xi_k^2+\xi_m^2}} & \imath \beta z_2 \xi_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Substituting this decomposition into (21) we get

$$\int_0^\infty G(\xi_k, \xi_m, z_1) G_1(\xi_k, \xi_m, z_2) e^{-2z_2\sqrt{\xi_k^2+\xi_m^2}} \begin{pmatrix} a_1(z_2, k, m) \\ b_1(z_2, k, m) \\ -\imath c_1(z_2, k, m) \end{pmatrix} dz_2$$
$$= \lambda_{k,m}^1 \begin{pmatrix} a_1(z_1, k, m) \\ b_1(z_1, k, m) \\ -\imath c_1(z_1, k, m) \end{pmatrix}.$$

Multiplying both sides of the last equation by

$$G^{-1}(\xi_x, \xi_y, z_1) = \mathrm{I} + \beta z_1 \begin{pmatrix} \frac{\xi_k^2}{\sqrt{\xi_k^2+\xi_m^2}} & \frac{\xi_k \xi_m}{\sqrt{\xi_k^2+\xi_m^2}} & \imath \xi_k \\ \frac{\xi_k \xi_m}{\sqrt{\xi_k^2+\xi_m^2}} & \frac{\xi_m^2}{\sqrt{\xi_k^2+\xi_m^2}} & \imath \xi_m \\ \imath \xi_k & \imath \xi_m & -\sqrt{\xi_k^2 + \xi_m^2} \end{pmatrix}, \tag{22}$$

and substituting the values of a_1, b_1, c_1 we arrive at

$$\int_0^\infty e^{-2z_2\sqrt{\xi_k^2+\xi_m^2}} \begin{pmatrix} 1 - 2\beta \xi_k^2/(z_2\sqrt{\xi_k^2 + \xi_m^2}) + 2\beta^2 \xi_k^2 z_2^2 \\ 0 \\ 0 \end{pmatrix} dz_2 = \lambda_{k,m}^1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

After integration we get the result

$$\lambda^1_{k,m} = \widetilde{\lambda}^1_{k,m} = \left(1 - \frac{(1-\beta)\beta\xi_k^2}{\xi_k^2+\xi_m^2}\right) \Big/ 2\sqrt{\xi_k^2+\xi_m^2}.$$

Analogous evaluation for the second and third series of eigen-vectors results in the desired formulae for $\lambda^i_{k,m}$, $i = 2, 3$. The proof of Lemma 1 is complete. □

The expansions given in Theorem 2 follow from the Lemma 1 and the splitting (18) and (19). □

4 General Horizontally Homogeneous and Isotropic Boundary Excitations

4.1 Homogeneous and Isotropic Excitations

Let us consider the boundary value problem (1) when **g** is a homogeneous zero mean Gaussian vector random field with a correlation matrix B_g. The relevant spectral tensor S_g is related to B_g by

$$B_g(\tau'_x, \tau'_y) = F[S_g] = \frac{1}{2\pi}\int_{R^2} e^{\imath(\tau'_x\xi_x+\tau'_y\xi_y)} S_g(\xi_x, \xi_y)\, d\xi_x\, d\xi_y, \tag{23}$$

$$S_g(\xi_x, \xi_y) = F^{-1}[B_g] = \frac{1}{2\pi}\int_{R^2} e^{-\imath(\tau'_x\xi_x+\tau'_y\xi_y)} B_g(\tau'_x, \tau'_y)\, d\tau'_x\, d\tau'_y, \tag{24}$$

where $\tau'_x = x'_1 - x'_2$ and $\tau'_y = y'_1 - y'_2$. From (4) and (2) we obtain

$$\begin{aligned} B_u(x_1, y_1; x_2, y_2) = \frac{1}{2\pi}\int_{R^2} d\xi_x\, d\xi_y \int_{R^4} K(x_1 - x'_2 - \tau'_x, y_1 - y'_2 - \tau'_y, z_1) \\ \times S_g(\xi_x, \xi_y) K(x_2 - x'_2, y_2 - y'_2, z_2) e^{\imath(\tau'_x\xi_x+\tau'_y\xi_y)}\, d\tau'_x\, d\tau'_y\, dx'_2\, dy'_2, \end{aligned}$$

or in the variables $w_x = x_1 - x'_2 - \tau'_x$, $w_y = y_1 - y'_2 - \tau'_y$,

$$\begin{aligned} B_u = \frac{1}{2\pi}\int_{R^2} d\xi_x\, d\xi_y \int_{R^4} K(w_x, w_y, z_1) e^{-\imath(w_x\xi_x+w_y\xi_y)}\, dw_x\, dw_y \\ \times S_g(\xi_x, \xi_y) K(x_2 - x'_2, y_2 - y'_2, z_2) e^{\imath((x_1-x'_2)\xi_x+(y_1-y'_2)\xi_y)}\, dx'_2\, dy'_2. \end{aligned}$$

Now taking the new variables $q_x = x'_2 - x_2$, $q_y = y'_2 - y_2$ we get

$$\begin{aligned} B_u = \int_{R^2} d\xi_x\, d\xi_y\, F^{-1}[K](\xi_x, \xi_y, z_1) S_g(\xi_x, \xi_y) \\ \times \int_{R^2} F^{-1}[K](-q_x, -q_y, z_2) e^{-\imath(q_x\xi_x+q_y\xi_y)} e^{\imath(\xi_x(x_1-x_2)+\xi_y(y_1-y_2))}\, dq_x\, dq_y. \end{aligned}$$

Using the change of variables $\tau_x = x_1 - x_2$, $\tau_y = y_1 - y_2$ we arrive at

$$\begin{aligned} & B_u(\tau_x, \tau_y, z_1, z_2) \\ & = 2\pi \int_{R^2} F^{-1}[K](\xi_x, \xi_y, z_1) S_g(\xi_x, \xi_y) F^{-1}[K(-q_x, -q_y, z_2)] e^{\imath(\xi_x\tau_x+\xi_y\tau_y)}\, d\xi_x\, d\xi_y. \end{aligned}$$

From the last formula we see that the correlation tensor B_u depends on the differences $x_1 - x_2$, and $y_1 - y_2$, i.e. $\mathbf{u}$ is partially homogeneous. After taking the Fourier transform we get (see the proof of Theorem 1)

$$B_u(\tau_x, \tau_y, z_1, z_2) \\ = \frac{1}{2\pi}\int_{R^2} e^{-\sqrt{\xi_x^2+\xi_y^2}(z_1+z_2)} G(\xi_x, \xi_y, z_1) S_g(\xi_x, \xi_y) G^*(\xi_x, \xi_y, z_2) e^{\imath(\xi_x\tau_x+\xi_y\tau_y)}\, d\xi_x\, d\xi_y$$

hence the partial spectral tensor looks like

$$S_u(\xi_x, \xi_y, z_1, z_2) = e^{-\sqrt{\xi_x^2+\xi_y^2}(z_1+z_2)} G(\xi_x, \xi_y, z_1) S_g(\xi_x, \xi_y) G^*(\xi_x, \xi_y, z_2).$$

Note that if $\mathbf{g}$ is an isotropic field, then B_g depends only on $r = \sqrt{\tau_x^2 + \tau_y^2}$, and we will write in this case $B_g(r)$. Then, the relevant spectral representation simplifies to

$$S_g(\xi_x, \xi_y) = \int_0^\infty B_g(r') r' J_0(r'\rho)\, dr'. \tag{25}$$

Indeed, by the definition (24),

$$S_g = \frac{1}{2\pi}\int_{R^2} e^{-\imath(\xi_x\tau_x'+\xi_y\tau_y')} B_g(\tau_x', \tau_y')\, d\xi_x\, d\xi_y,$$

or in the polar coordinates $\tau_x' = r'\cos\varphi$, $\tau_y' = r'\sin\varphi$ and $\xi_x = \rho\cos\psi$, $\xi_y = \rho\sin\psi$

$$S_g = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-\imath r'\rho\cos(\varphi-\psi)} B_g(r', \varphi) r'\, dr'\, d\varphi.$$

If $\mathbf{g}$ is isotropic, the tensor B_g does not depend on the angle φ, hence

$$S_g = \frac{1}{2\pi}\int_0^\infty B_g(r') r' \int_0^{2\pi} e^{-\imath r'\rho\cos(\varphi-\psi)}\, d\varphi\, dr'.$$

The inner integral is the Bessel function $J_0(r'\rho)$, and we get the representation (25). Then,

$$B_u(\tau_x, \tau_y, z_1, z_2) = \int_0^\infty \int_0^{2\pi} e^{-\rho(z_1+z_2)} G(\rho, \psi, z_1) \int_0^\infty B_g(r') r' J_0(r'\rho)\, dr' \\ \times G^*(\rho, \psi, z_2) e^{\imath\rho(\cos\psi\,\tau_x+\sin\psi\,\tau_y)} \rho d\rho\, d\psi. \tag{26}$$

Notice that in the case of a white noise

$$B_g(r')_{ij} = \frac{\delta_{ij}\delta(r')}{2\pi r'},$$

so

$$B_u(\tau_x, \tau_y, z_1, z_2) \\ = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-\rho(z_1+z_2)} G(\rho, \psi, z_1) G^*(\rho, \psi, z_2) e^{\imath\rho(\cos\psi\,\tau_x+\sin\psi\,\tau_y)} \rho d\rho\, d\psi.$$

In polar coordinates, $\tau_x = R_\tau \cos\phi$, $\tau_y = R_\tau \sin\phi$, hence

$$B_u(R_\tau, \phi, z_1, z_2) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-\rho(z_1+z_2)} G(\rho, \psi, z_1) G^*(\rho, \psi, z_2) e^{\imath\rho R_\tau \cos(\psi-\phi)} \rho d\rho\, d\psi.$$

The entries $\{B_u\}_{ij}$, $i, j = 1, 2, 3$ can be evaluated explicitly by the formulae (40)–(51), from Appendix B, and we get the same representation as in Theorem 1.

In the general case, to express B_u through B_g, it is convenient to introduce a new notation $\hat{B}_u$ and $\hat{B}_g$ by arranging the entries of the correlation tensor in a 9-dimensional column vector. Then the representation (26) is conveniently written as

$$\hat{B}_u(R_\tau, \phi, z_1, z_2) = \int_0^\infty A(R_\tau, \phi, z_1, z_2, r') \hat{B}_g(r')\, dr'$$

where

$$A(R_\tau, \phi, z_1, z_2, r')_{9\times 9} = \frac{1}{2\pi}\int_{R^2} e^{-\rho(z_1+z_2)} G(\rho, \psi, z_1) \otimes G^*(\rho, \psi, z_2) e^{\imath\rho R_\tau \cos(\psi-\phi)} J_0(r'\rho) \rho d\rho\, d\psi.$$

Here we denote by $\otimes$ a tensor product of two matrices. The entries A_{ij} can be evaluated explicitly, as it was done in the case of the matrix B_u. In the next section we present an example.

4.2 An Example of Boundary Excitations with Finite Correlation Length

In this section we analyze an example with boundary excitations having a finite correlation length.

So let us consider the boundary problem (1) where the isotropic Gaussian random field $\mathbf{g}$ is defined by the following spectral tensor

$$S_g(\xi_x, \xi_y) = \mathrm{I} L e^{-\rho L}, \quad \rho = \sqrt{\xi_x^2 + \xi_y^2} \tag{27}$$

where I is an identity matrix, and L is the correlation lengths of g_i. Then,

$$B_u(R_\tau, \phi, z_1, z_2) = \frac{L}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-\rho(z_1+z_2+L)} G(\rho, \psi, z_1) G^*(\rho, \psi, z_2) e^{\imath\rho R_\tau \cos(\psi-\phi)} \rho d\rho\, d\psi.$$

Using the formulae (40)–(51) presented in Appendix B we carry out the integration explicitly. This yields

$$B_u = \frac{L}{2\pi r_L^3}\Bigg\{ [(1-\beta)\bar{z} + L]\mathrm{I}$$

$$+ \frac{3\beta}{r_L^2}\begin{pmatrix} \tau_x^2 \bar{z} & \tau_x\tau_y\bar{z} & \tau_x(z_1-z_2)(\bar{z}+L) \\ \tau_x\tau_y\bar{z} & \tau_y^2\bar{z} & \tau_y(z_1-z_2)(\bar{z}+L) \\ \tau_x(z_1-z_2)(\bar{z}+L) & \tau_y(z_1-z_2)(\bar{z}+L) & \bar{z}(\bar{z}+L)^2 \end{pmatrix}$$

$$+\frac{6\beta^2 z_1 z_2}{r_L^4}$$
$$\times\begin{pmatrix} (r_L^2-5\tau_x^2)(\overline{z}+L) & -5\tau_x\tau_y(\overline{z}+L) & \tau_x(5(\overline{z}+L)^2-r_L^2) \\ -5\tau_x\tau_y(\overline{z}+L) & (r_L^2-5\tau_y^2)(\overline{z}+L) & \tau_y(5(\overline{z}+L)^2-r_L^2) \\ \tau_x(r_L^2-5(\overline{z}+L)^2) & \tau_y(r_L^2-5(\overline{z}+L)^2) & (\overline{z}+L)(2r_L^2-5(\tau_x^2+\tau_y^2)) \end{pmatrix}\Bigg\}, \tag{28}$$

where

$$r_L=\sqrt{\tau_x^2+\tau_y^2+(z_1+z_2+L)^2},\quad \overline{z}=z_1+z_2.$$

The Karhunen-Loève expansion in this case takes the form

$$\mathbf{u}(x,y,z)=\frac{1}{2\sqrt{R_1R_2}}\sum_{\substack{k,m=-\infty\\(k,m)\neq(0,0)}}^{\infty} e^{-\pi(z+L/2)\sqrt{(k/R_1)^2+(m/R_2)^2}}$$
$$\times\Big[\big(\Re G(k,m,z)\boldsymbol{\zeta}_{k,m}+\Im G(k,m,z)\boldsymbol{\eta}_{k,m}\big)\cos\pi\left(\frac{kx}{R_1}+\frac{my}{R_2}\right)$$
$$+\big(\Re G(k,m,z)\boldsymbol{\eta}_{k,m}-\Im G(k,m,z)\boldsymbol{\zeta}_{k,m}\big)\sin\pi\left(\frac{kx}{R_1}+\frac{my}{R_2}\right)\Big] \tag{29}$$

where $\boldsymbol{\eta}_{k,m}$, $\boldsymbol{\zeta}_{k,m}$ are families of independent standard Gaussian vectors, and $G(k,m,z)$ is defined in (14). The relevant series expansion for the correlation tensor B_u is

$$B_u(\tau_x,\tau_y,z_1,z_2)=\frac{1}{4R_1R_2}\sum_{\substack{k,m=-\infty\\(k,m)\neq(0,0)}}^{\infty} e^{-\pi(z_1+z_2+L)\sqrt{(k/R_1)^2+(m/R_2)^2}}$$
$$\times\Big[\Re\{G(k,m,z)G^*(k,m,z)\}\cos\pi\left(\frac{k\tau_x}{R_1}+\frac{m\tau_y}{R_2}\right)$$
$$-\Im\{G(k,m,z)G^*(k,m,z)\}\sin\pi\left(\frac{k\tau_x}{R_1}+\frac{m\tau_y}{R_2}\right)\Big]. \tag{30}$$

5 Simulation Results

Let us present some simulation results for two cases: (1) the boundary excitations are produced by a 2D white noise, (2) the boundary displacements have a finite correlation length. The white noise case is validated by a comparison of calculations carried out according to the K-L expansion (17) with the exact results (5). In the case of finite correlation L we compare the correlation functions calculated according to the K-L series expansion (30) and the direct Monte Carlo simulations based on (29) against the exact result (28). In all calculations however the Monte Carlo errors were smaller than 0.5% so that the exact and plotted curves are undistinguishable on the graph. Therefore we present only the exact and series based calculation results.

In Fig. 1 (left panel) the correlations B_{ij} as functions of the longitudinal increment $x=x_1-x_2$ are plotted for the case of white noise excitations, for fixed values of $y_1=y_2=1$

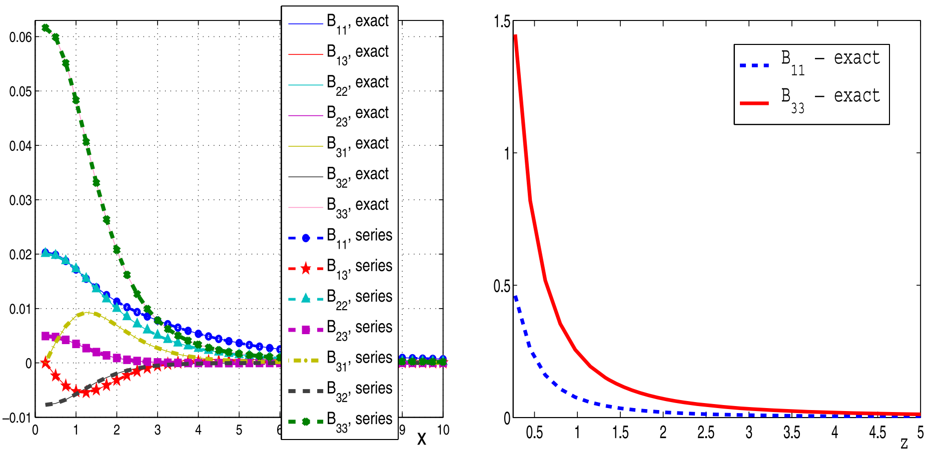

Fig. 1 (Color online) The case of white noise excitations. *Left panel*: The correlation functions B_{ij} versus the horizontal coordinate $x = x_1 - x_2$, for fixed $y_1 = y_2 = 1$ and $z_1 = z_2 = 1$. Compared are the exact results with the series expansions. The elasticity constant was fixed as $\alpha = 2$, the number of harmonics in the K-L expansion was $k = m = 200$, and the cut-off parameters R_1, R_2 were taken as $R_1 = R_2 = 100$. *Right panel*: The correlations functions B_{11} and B_{33} versus the coordinate $z = z_2$ (at $z_1 = 0.1$), for the same fixed values of x_1, x_2, y_1, y_2

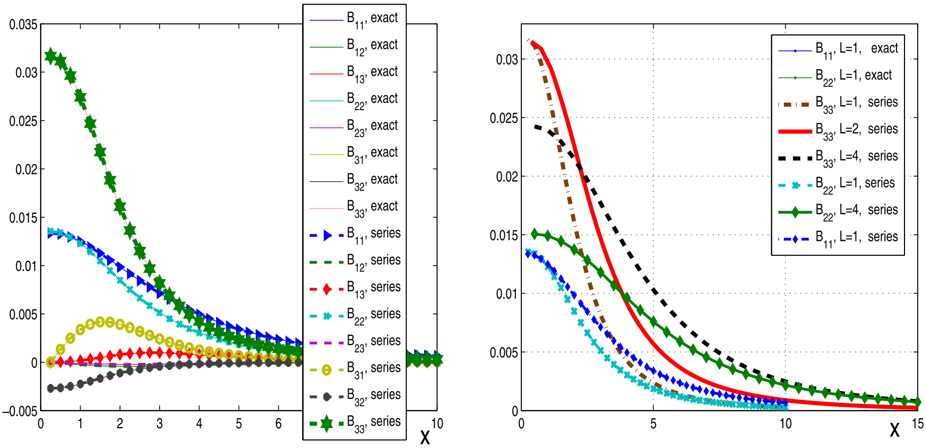

Fig. 2 (Color online) The case of finite correlation length L. *Left panel*: the correlation functions B_{ij} versus the horizontal coordinate $x = x_1 - x_2$, for fixed $y_1 = y_2$ and $z_1 = z_2 = 1$, exact solutions and series representations. *Right panel*: The correlation function B_{33} is shown for three different values of the correlation length L, $L = 1, 2$ and 4, the correlation function B_{22}, for $L = 1$, and $L = 4$, and the correlation function B_{11}, for $L = 1$. All functions are shown both exact and as series expansions; other parameters the same as in the *left panel*

and $z_1 = z_2 = 1$. Here we compare the K-L expansion (17) and the exact result (5), showing an excellent agreement. In the right panel of Fig. 1 we just show the exact results for the correlations B_{11} and B_{33} as functions of the vertical coordinate.

The results for the case of finite correlations are presented in Fig. 2. In this figure (left panel) we plot all the correlation and cross-correlation functions versus the longitudinal

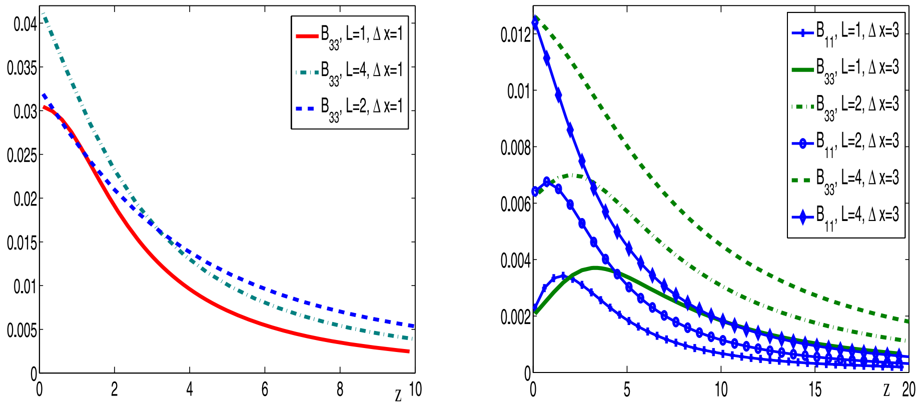

Fig. 3 (Color online) The case of finite correlation length L. *Left panel*: the correlation function B_{33} versus the vertical coordinate z (for fixed $\Delta x = x_1 - x_2 = y_1 - y_2 = 1$, exact solutions) plotted for three different values of the correlation length L. *Right panel*: The correlation functions B_{33} and B_{11} versus the vertical coordinate are plotted for three different values of the correlation length L, $L = 1, 2$ and 4, for fixed $\Delta x = 3$

coordinate, also for fixed values of $y_1 = y_2$ and $z_1 = z_2 = 1$, for $L = 1$. Here both the exact results and the series-based calculations are shown which are practically coincident. To show the impact of the correlation length L of the boundary input excitations, we present in Fig. 2 (right panel) the correlation function $B_{33}(x)$ for $L = 1, 2$ and $L = 4$. It is clearly seen that with the increase of the correlation length L the correlation function B_{33} gets heavier tails, but as to the intensity of fluctuations at small values of x, there is no monotone behaviour. So for small values of x, the correlations at $L = 1$ and $L = 2$ are almost the same, while for $L = 4$ the correlation is considerably less. It is interesting to notice that for B_{22} the situation is converse: the correlations for $L = 4$ are larger than those at $L = 1$. Note also that the correlation functions B_{11} and B_{22} for $L = 1$ are close at small distances, and get closer after $x = 10$ while in between, there is a clear difference.

Further interesting issues are: how propagate the boundary excitations in the vertical direction, and what is the influence of the input correlation length L. In Fig. 3 (left panel) we show the vertical profile of the correlation function B_{33} for three different values of L, for fixed values $\Delta x = x_1 - x_2 = y_1 - y_2 = 1$. All three curves are monotonically decreasing with the height, but notice that the case $L = 1 = \Delta x$ is a bit different, having a Gaussian type behaviour at small heights. This behaviour is more pronounced for larger values of Δx, see the right panel of Fig. 3 where we plot the correlations B_{33} and B_{11} for $\Delta x = 3$. Here we see that for the cases when $L < \Delta x$ these correlations first increase, reaching a maximum value, and then decrease, while they are monotonically decreasing if $L > \Delta x$. For larger value of Δx, the non-monotonic behaviour is more pronounced, see Fig. 4 (left panel). Finally, the impact of the correlation length L on the cross-correlations is seen from the results presented in the right panel of Fig. 4.

To conclude we remark that all these numerical results serve as illustrations and validations of the derived exact representations. As to the choice of an efficient Mote Carlo simulation of the random solution, we would suggest to use a stratified version of the Randomized spectral method or, if one wishes to have samples with good ergodic properties, the Fourier-wavelet expansion can be constructed (for details see [10]).

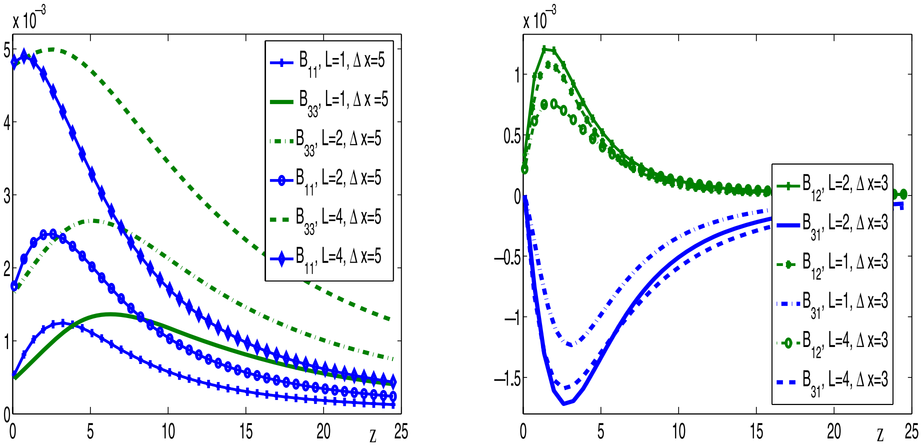

Fig. 4 (Color online) The case of finite correlation length L. *Left panel*: the same as in Fig. 3, right panel, but for $\Delta x = 5$. *Right panel*: the same as in Fig. 3, right panel, but for the cross-correlation functions

Appendix A

The Fourier transforms of function $g(x, y, \cdot)$ over the variables x, y are defined by

$$G(\xi_x, \xi_y, \cdot) = F^{-1}[g(x, y, \cdot)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\imath(x\xi_x + y\xi_y)} g(x, y, \cdot)\, dx\, dy,$$

and

$$g(x, y, \cdot) = F[G(\xi_x, \xi_y, \cdot)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\imath(x\xi_x + y\xi_y)} G(\xi_x, \xi_y, \cdot)\, d\xi_x\, d\xi_y.$$

We use the simple property of the Fourier transformation

$$\begin{aligned} F^{-1}[D^{\alpha+\gamma}_{x,y} g(x, y, z)] &= (\imath\xi)^{\alpha+\gamma} F^{-1}[g(x, y, z)], \\ F^{-1}[D^{\alpha}_{z} g(x, y, z)] &= D^{\alpha}_{z} F^{-1}[g(x, y, z)]. \end{aligned} \tag{31}$$

It is known that (see [11])

$$I_0 = F^{-1}\left[\frac{1}{r}\right] = \frac{1}{\sqrt{\xi_x^2 + \xi_y^2}} e^{-z\sqrt{\xi_x^2+\xi_y^2}}, \quad r = \sqrt{x^2 + y^2 + z^2},$$

where x and y are the variables of the Fourier transform, and z is a free variable. Taking the derivatives it is easy to find that

$$F^{-1}\left[\frac{z}{r^3}\right] = -F^{-1}\left[\frac{\partial I_0}{\partial z}\right] = e^{-z\sqrt{\xi_x^2+\xi_y^2}}, \tag{32}$$

and

$$F^{-1}\left[\frac{zx^2}{r^5}\right] = \frac{1}{3} \frac{\partial^2}{\partial \xi_x^2}\left(\frac{\partial I_0}{\partial z}\right) = \frac{1}{3}\left(e^{-z\sqrt{\xi_x^2+\xi_y^2}} - \frac{z\xi_x^2}{\sqrt{\xi_x^2 + \xi_y^2}} e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right). \tag{33}$$

We can rewrite the last formula as follows

$$-\frac{z\xi_x^2}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}} = 3F^{-1}\left[\frac{x^2z}{r^5}\right] - F^{-1}\left[\frac{z}{r^3}\right],$$

a similar formula for the variable y

$$-\frac{z\xi_y^2}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}} = 3F^{-1}\left[\frac{zy^2}{r^5}\right] - F^{-1}\left[\frac{z}{r^3}\right],$$

and for the product xy

$$F^{-1}\left[\frac{xy}{r^5}\right] = \frac{1}{3}F^{-1}\left[\frac{\partial^2}{\partial x\partial y}\frac{1}{r}\right] = -\frac{1}{3}\frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}}.$$

By the property (31) we get

$$-\imath\xi_x e^{-z\sqrt{\xi_x^2+\xi_y^2}} = 3F^{-1}\left[\frac{zx}{r^5}\right], \qquad -\imath\xi_y e^{-z\sqrt{\xi_x^2+\xi_y^2}} = 3F^{-1}\left[\frac{zy}{r^5}\right], \tag{34}$$

$$F^{-1}\left[\frac{z^3}{r^5}\right] = -\frac{z^2}{3}F^{-1}\left[\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)\right] = \frac{1}{3}\left(e^{-z\sqrt{\xi_x^2+\xi_y^2}} + z\sqrt{\xi_x^2+\xi_y^2}e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right), \tag{35}$$

or

$$z\sqrt{\xi_x^2+\xi_y^2}e^{-z\sqrt{\xi_x^2+\xi_y^2}} = 3F^{-1}\left[\frac{z^3}{r^5}\right] - F^{-1}\left[\frac{z}{r^3}\right].$$

We need also the next representations

$$\xi_x^2 e^{-z\sqrt{\xi_x^2+\xi_y^2}} = -\frac{\partial}{\partial z}\left[\frac{\xi_x^2}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right] = -F^{-1}\left[\frac{15zx^2}{r^7} - \frac{3z}{r^5}\right]. \tag{36}$$

Here and in what follows we write z instead of z_1+z_2,

$$\xi_y^2 e^{-z\sqrt{\xi_x^2+\xi_y^2}} = -\frac{\partial}{\partial z}\left[\frac{\xi_y^2}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right] = -F^{-1}\left[\frac{15zy^2}{r^7} - \frac{3z}{r^5}\right], \tag{37}$$

$$\xi_x\xi_y e^{-z\sqrt{\xi_x^2+\xi_y^2}} = -\frac{\partial}{\partial z}\left(\frac{\xi_x\xi_y}{\sqrt{\xi_x^2+\xi_y^2}}e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right) = -15F^{-1}\left[\frac{zxy}{r^7}\right], \tag{38}$$

and finally, the last group

$$\begin{aligned} -\imath\xi_x\sqrt{\xi_x^2+\xi_y^2}e^{-z\sqrt{\xi_x^2+\xi_y^2}} &= \frac{\partial}{\partial z}\left(\imath\xi_x e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right) = -F^{-1}\left[\frac{3x}{r^5} - \frac{15z^2x}{r^7}\right], \\ -\imath\xi_y\sqrt{\xi_x^2+\xi_y^2}e^{-z\sqrt{\xi_x^2+\xi_y^2}} &= \frac{\partial}{\partial z}\left(\imath\xi_y e^{-z\sqrt{\xi_x^2+\xi_y^2}}\right) = -F^{-1}\left[\frac{3y}{r^5} - \frac{15z^2y}{r^7}\right]. \end{aligned} \tag{39}$$

Appendix B

The following formulae are obtained by using the known integral cited in [5]:

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos(\varphi-\phi)} e^{-\rho z} \rho \, d\varphi \, d\rho = \int_0^\infty e^{-\rho z} J_0(\rho R_\tau)\rho \, d\rho = \frac{z}{(z^2+R_\tau^2)^{3/2}}. \tag{40}$$

To evaluate the integral

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos(\varphi-\phi)} e^{-\rho z} \xi_x^2 \, d\varphi \, d\rho \bigg|_{\xi_x=\rho\cos\varphi}$$

we use the change of variables $\varphi' = \varphi - \phi$, so that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \int_{-\phi}^{2\pi-\phi} e^{\imath R_\tau \rho \cos\varphi'} e^{-\rho z} \rho^2 (\cos^2\varphi' \cos^2\phi + \sin^2\varphi' \sin^2\phi) d\varphi' \, d\rho \\
&= \int_0^\infty e^{-\rho z}\left[\left(\frac{J_1(\rho R_\tau)}{\rho R_\tau} - J_2(\rho R_\tau)\right)\cos^2\phi + \frac{J_1(\rho R_\tau)}{\rho R_\tau}\sin^2\phi\right]\rho^2 \, d\rho \\
&= \frac{1}{(z^2+R_\tau^2)^{3/2}}\left(1 - \frac{3R_\tau^2\cos^2\phi}{z^2+R_\tau^2}\right) = \frac{1}{(z^2+R_\tau^2)^{3/2}}\left(1 - \frac{3\tau_x^2}{z^2+R_\tau^2}\right).
\end{aligned} \tag{41}$$

Similar calculation for $(B_u)_{22}$ yields

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos(\varphi-\phi)} e^{-\rho z} \xi_y^2 \, d\varphi \, d\rho \bigg|_{\xi_y=\rho\sin\varphi, \varphi'=\varphi-\phi} \\
&= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos\varphi'} e^{-\rho z} \rho^2 (\cos^2\varphi' \sin^2\phi + \sin^2\varphi' \cos^2\phi) d\varphi' d\rho \\
&= \int_0^\infty e^{-\rho z}\left[\left(\frac{J_1(\rho R_\tau)}{\rho R_\tau} - J_2(\rho R_\tau)\right)\sin^2\phi + \frac{J_1(\rho R_\tau)}{\rho R_\tau}\cos^2\phi\right]\rho^2 \, d\rho \\
&= \frac{1}{(z^2+R_\tau^2)^{3/2}}\left(1 - \frac{3R_\tau^2\sin^2\phi}{z^2+R_\tau^2}\right) = \frac{1}{(z^2+R_\tau^2)^{3/2}}\left(1 - \frac{3\tau_y^2}{z^2+R_\tau^2}\right).
\end{aligned} \tag{42}$$

Further, for B_{u12} we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos(\varphi-\phi)} e^{-\rho z} \xi_x \xi_y \, d\varphi \, d\rho \bigg|_{\xi_x=\rho\cos\varphi, \xi_y=\rho\sin\varphi} \\
&= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{\imath R_\tau \rho \cos\varphi'} e^{-\rho z} \cos\varphi \sin\varphi \, d\varphi' \, d\rho \\
&= -\int_0^\infty e^{-\rho z} J_2(\rho R_\tau)\cos\phi \sin\phi \rho^2 \, d\rho \\
&= -\frac{3\tau_x \tau_y}{(z^2+R_\tau^2)^{5/2}}.
\end{aligned} \tag{43}$$

For the last element $(B_u)_{13}$ we get

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_x \, d\varphi\,\rho\, d\rho\Big|_{\xi_x=\rho\cos\varphi}\\
&\quad=\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau\rho\cos\varphi'} e^{-\rho z}\rho^2\cos\varphi'\cos\phi d\varphi'\, d\rho\\
&\quad=\imath\int_0^\infty e^{-\rho z} J_1(\rho R_\tau)\cos\phi\rho^2\, d\rho \ = \frac{3R_\tau z\imath}{(z^2+R_\tau^2)^{5/2}}\cos\phi = \frac{3z\imath\tau_x}{(z^2+R_\tau^2)^{5/2}},
\end{aligned} \tag{44}
$$

$$
\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\rho^2\, d\varphi\, d\rho = \int_0^\infty e^{-\rho z} J_0(\rho R_\tau)\rho^2\, d\rho = \frac{2z^2-R_\tau^2}{(z^2+R_\tau^2)^{5/2}}, \tag{45}
$$

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_x^2 \, d\varphi\,\rho d\rho\Big|_{\xi_x=\rho\cos\varphi,\,\varphi'=\varphi-\phi}\\
&\quad=\int_0^\infty e^{-\rho z}\left(\frac{J_1(\rho R_\tau)}{\rho R_\tau} - J_2(\rho R_\tau)\cos^2\phi\right)\rho^3\, d\rho = \frac{3z}{(z^2+R_\tau^2)^{5/2}}\left(1-\frac{5\tau_x^2}{z^2+R_\tau^2}\right),
\end{aligned} \tag{46}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_y^2 \, d\varphi\,\rho d\rho\Big|_{\xi_y=\rho\sin\varphi,\varphi'=\varphi-\phi}\\
&\quad=\int_0^\infty e^{-\rho z}\left(\frac{J_1(\rho R_\tau)}{\rho R_\tau} - J_2(\rho R_\tau)\sin^2\phi\right)\rho^3\, d\rho = \frac{3z}{(z^2+R_\tau^2)^{5/2}}\left(1-\frac{5\tau_y^2}{z^2+R_\tau^2}\right),
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_x\xi_y \, d\varphi\,\rho d\rho\\
&\quad=-\int_0^\infty e^{-\rho z} J_2(\rho R_\tau)\cos\phi\sin\phi\rho^3\, d\rho = -z\frac{15\tau_x\tau_y}{(z^2+R_\tau^2)^{7/2}},
\end{aligned} \tag{48}
$$

$$
\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\, d\varphi\,\rho^3 d\rho = \int_0^\infty e^{-\rho z} J_0(\rho R_\tau)\rho^3\, d\rho = 3z\frac{2z^2-3R_\tau^2}{(z^2+R_\tau^2)^{7/2}}, \tag{49}
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_x \, d\varphi\,\rho^2\, d\rho\Big|_{\xi_x=\rho\cos\varphi}\\
&\quad=\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau\rho\cos\varphi'} e^{-\rho z}\rho^3\cos\varphi'\cos\phi d\varphi'\, d\rho\\
&\quad=\imath\int_0^\infty e^{-\rho z} J_1(\rho R_\tau)\cos\phi\rho^3\, d\rho \ = \imath\frac{3\tau_x(4z^2-R_\tau^2)}{(z^2+R_\tau^2)^{7/2}},
\end{aligned} \tag{50}
$$

$$
\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} e^{\imath R_\tau \rho\cos(\varphi-\phi)} e^{-\rho z}\xi_y \, d\varphi\,\rho^2\, d\rho\Big|_{\xi_y=\rho\sin\varphi} = \imath\frac{3\tau_y(4z^2-R_\tau^2)}{(z^2+R_\tau^2)^{7/2}}. \tag{51}
$$

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